AN APPLICATION OF SEVERI'S THEORY OF A BASIS TO THE KUMMER AND WEDDLE SURFACES*

BY

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In a series of papers F. Severi† has developed a theory by means of which any linear system of algebraic curves on an algebraic surface can be expressed linearly in terms of a finite number of such systems. He has also applied this theory to the study of the curves on the Fano quartic surface and to the determination of the group of birational transformations which leave this surface invariant.

In the following paper this method is applied to the study of certain of the birational transformations which leave the Kummer and Weddle surfaces invariant. In particular the question of the periodicity of the product of two such transformations is studied. Some new relations among the different types of transformation are obtained.

1. The Kummer Surface

Let A, B, E, F be any Göpel even tetrad of nodes on the general Kummer surface K_4. The notation‡

$$(1) \qquad \begin{matrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{matrix}$$

will be used to denote the complete configuration of the sixteen nodes on K_4. Variables x, y, z, w will be so chosen that $A \equiv (1, 0, 0, 0)$, $B \equiv (0, 1, 0, 0)$, $E \equiv (0, 0, 1, 0)$ and $F \equiv (0, 0, 0, 1)$.

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† *Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica*, Mathematische Annalen, vol. 62 (1906), pp. 194–225; *La base minima pour la totalité des courbes tracées sur une surface algébrique*, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), pp. 449–468; *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

‡ See Hudson, *Kummer's Quartic Surface*, Chapters I, II, and VII.

The surface K_4, referred to this system of coördinates, is invariant under the cubic inversion*

$$x' : y' : z' : w' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{w}. \tag{2}$$

Under this transformation the planes of each pencil through an edge of the tetraedron of inversion are interchanged in pairs. Through each pair of vertices are two six-point conics, namely:

$$\begin{cases} ABDGKO, \\ ABCHLP, \end{cases} \quad \begin{cases} AEJKLM, \\ AEIONP, \end{cases} \quad \begin{cases} AFCDJN, \\ AFGHIM, \end{cases}$$

$$\begin{cases} EFCHKO, \\ EFDGLP, \end{cases} \quad \begin{cases} BFIKLN, \\ BFJMOP, \end{cases} \quad \begin{cases} BECDIM, \\ BEGHJN. \end{cases}$$

Hence these pairs of conics are interchanged under (2). This requires that twelve nodes interchange in pairs as follows

$$(CG)\,(DH)\,(IJ)\,(KP)\,(LO)\,(MN), \tag{3}$$

where (CG) indicates the interchange of the points C and G. It follows that each of the four remaining conics $CGIJLO$, $DHLOMN$, $DHIJKP$, and $CGKPMN$, which are plane sections of K_4 by the same quadric, is invariant under (2). Since K_4 is invariant and each of the vertices A, B, E, F is transformed into the opposite face of the tetraedron of inversion by (2), the image on K_4 of each vertex is the rational plane section of the surface by the opposite face. Denote a general plane section of K_4 by C_4 and use the symbol $\backsim$ to denote "is transformed into." The transformations of the vertices are

$$\begin{aligned} A &\backsim C_4 - B - E - F, \qquad & B &\backsim C_4 - A - E - F, \\ E &\backsim C_4 - A - B - E, \qquad & F &\backsim C_4 - A - B - E, \end{aligned} \tag{4}$$

respectively.† By (2) any plane is transformed into a cubic surface which contains the edges of the tetraedron $ABEF$. This surface intersects K_4 in C_{12}, a curve of order 12. Since genus is invariant under birational transformation and C_{12} is the image of any plane section of K_4, the genus of this curve is 3. The inversion is also symmetric with respect to all the vertices: hence

$$C_4 \backsim 3C_4 - 2(A + B + E + F). \tag{5}$$

* See J. I. Hutchinson, *On some birational transformations of the Kummer surface into itself*, Bulletin of the American Mathematical Society, ser. 2, vol. 7 (1901), pp. 211–217; see p. 212.

† The symbol $\lambda C_4 - \alpha A - \beta B - \cdots$ denotes the complete curve of intersection of the Kummer surface with a surface of order λ. This curve of intersection passes through the points A, B, $\cdots$, 2α, 2β, $\cdots$, times respectively, is of order 4λ, and is of genus $1 + 2\lambda^2 - \alpha^2 - \beta^2 - \cdots$. For details as to the meaning of symbolic addition and subtraction see Picard et Simart, *Fonctions algébriques de deux variables*, vol. 2, pp. 104–116 or the papers of Severi already cited.

The partial transformations (3), (4), and (5) completely* express the transformation of the linear systems of the curves on K_4 under the inversion (2). This complete transformation will be denoted by $[{}^{AB}_{EF}]$. The inversion as thus expressed is geometric in character and independent of the coördinate system.

The inversion with respect to any other Göpel even tetrad of nodes can readily be written. The transformation $[{}^{AB}_{IJ}]$ is

$$(CK)\,(DL)\,(EF)\,(GP)\,(HO)\,(MN),$$

$$A \backsim C_4 - B - I - J, \quad B \backsim C_4 - A - I - J, \quad I \backsim C_4 - A - B - J,$$

$$J \backsim C_4 - A - B - I, \quad C_4 \backsim 3C_4 - 2(A + B + I + J);$$

and $[{}^{AB}_{MN}]$ is

$$(CO)\,(DP)\,(EF)\,(GL)\,(HK)\,(IJ),$$

$$A \backsim C_4 - B - M - N, \quad B \backsim C_4 - A - M - N,$$

$$M \backsim C_4 - A - B - N, \quad N \backsim C_4 - A - B - M,$$

$$C_4 \backsim 3C_4 - 2(A + B + M + N).$$

By the symbol $[{}^{AB}_{EF}]\,[{}^{AB}_{IJ}]$, referred to as a product, is meant that a basis (α) of elements is transformed into a configuration (β) by $[{}^{AB}_{EF}]$ and that (β) is transformed into (γ) by $[{}^{AB}_{IJ}]$. The product transforms (α) into (γ). Thus $[{}^{AB}_{EF}]\,[{}^{AB}_{IJ}]$ is

$$A \backsim 2C_4 - A - 2B - E - F - I - J,$$

$$B \backsim 2C_4 - 2A - B - E - F - I - J,$$

$$E \backsim C_4 - A - B - E, \quad F \backsim C_4 - A - B - F, \quad I \backsim C_4 - A - B - I,$$

$$J \backsim C_4 - A - B - J, \quad C_4 \backsim 5C_4 - 4(A + B) - 2(E + F + I + J),$$

$$(CP)\,(DO)\,(GK)\,(HL)\,(MM)\,(NN).$$

The symmetry of this result in E, F, I, and J shows that the transformation is of period two.† Similarly the products $[{}^{AB}_{EF}]\,[{}^{AB}_{MN}]$, $[{}^{AB}_{IJ}]\,[{}^{AB}_{MN}]$ and

* See Severi, Annales de l'école normale supérieure, ser. 3, vol. 25 (1908), p. 465.

† See H. F. Baker, *Note on some transformations of a general Kummer surface*, Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912–13), pp. 302–312, where this result is obtained by a different method. The operations in Baker's paper may be identified with those of the present paper by the following key:

$$\omega \equiv \begin{bmatrix} AB \\ EF \end{bmatrix}, \quad \omega U_{DA} \equiv \begin{bmatrix} AB \\ IJ \end{bmatrix}, \quad \omega V_{DA} \equiv \begin{bmatrix} AB \\ MN \end{bmatrix},$$

and

$$\vartheta_2\,\vartheta_3, \quad \vartheta_3\,\vartheta_1, \quad \vartheta_1\,\vartheta_2$$

with

$$(AF)\,(BE)\,(CH)\,(DG)\,(IN)\,(JM)\,(KP)\,(LO),$$
$$(AE)\,(BF)\,(CG)\,(DH)\,(IM)\,(JN)\,(KO)\,(LP),$$
$$(AB)\,(CD)\,(EF)\,(GH)\,(IJ)\,(KL)\,(MN)\,(OP),$$

C_4 remaining invariant under each collineation.

$[{}^{AB}_{EF}][{}^{AB}_{IJ}][{}^{AB}_{MN}]$ are each of period two. Hence *three cubic inversions, taken with respect to Göpel even tetraedra which have an edge in common, generate an abelian group of order eight and type* (1, 1, 1). There are 120 such edges and hence 120 such groups.

The tetraedron $ACIK$ has a vertex in common with the tetraedron $ABEF$. The product $[{}^{AB}_{EF}][{}^{AC}_{IK}]$ is

$$
\begin{aligned}
&A \backsim 3C_4 - 2A - 2(C+I+K) - (G+J+P),\\
&B \backsim 2C_4 - 2A - (C+I+K) - (G+P),\\
&E \backsim 2C_4 - 2A - (C+I+K) - (J+P),\\
&F \backsim 2C_4 - 2A - (C+I+K) - (G+J),\\
&G \backsim C_4 - A - (I+K), \qquad C \backsim E,\\
&J \backsim C_4 - A - (C+K), \qquad I \backsim B,\\
&P \backsim C_4 - A - (C+I), \qquad K \backsim F,\\
&C_4 \backsim 7C_4 - 6A - 4(C+I+K) - 2(G+J+P),\\
&\qquad (DNO)(HLM),
\end{aligned} \tag{6}
$$

where (DNO) is the cyclic substitution on its elements. If we write

$$\alpha = C + I + K, \qquad \beta = G + J + P, \qquad \gamma = B + E + F,$$

then (6) in part becomes

$$
\begin{aligned}
&C_4 \backsim 7C_4 - 6A - 4\alpha - 2\beta, \qquad A \backsim 3C_4 - 2A - 2\alpha - \beta, \qquad \alpha \backsim \gamma,\\
&\beta \backsim 3C_4 - 3A - 2\alpha, \qquad \gamma \backsim 6C_4 - 6A - 3\alpha - 2\beta.
\end{aligned} \tag{7}
$$

If (7) is non-periodic, (6) can not be periodic. Transform (7) through

$$X = C_4 - 2A, \qquad Y = A - \beta, \qquad Z = \alpha - 2\beta + \gamma,$$
$$W = A - \alpha, \qquad T = \alpha,$$

and it becomes

$$X \backsim X, \qquad Y \backsim Y, \qquad Z \backsim Z, \qquad W \backsim 3X + 3Y - Z + W,$$
$$T \backsim -2Y + Z + 2W + T,$$

which is clearly non-periodic. Hence (6) is non-periodic. By suitable choice of variables the periodicity of the transformation given by the product of any two inversions, taken with respect to Göpel even tetraedra which have one node in common, can always be made to depend on the periodicity of (7). Hence the products of this type are all non-periodic.

Consider next the product $[{AB \atop EF}][{IJ \atop MN}]$ by which

$$
(8)\quad
\begin{gathered}
A \backsim 3C_4 - (A + E + F) - 2(I + J + M + N),\\
B \backsim 3C_4 - (B + E + F) - 2(I + J + M + N),\\
E \backsim 3C_4 - (A + B + E) - 2(I + J + M + N),\\
F \backsim 3C_4 - (A + B + F) - 2(I + J + M + N),\\
I \backsim C_4 - (I + M + N), \qquad J \backsim C_4 - (J + M + N),\\
M \backsim C_4 - (I + J + M), \qquad N \backsim C_4 - (I + J + N),\\
C_4 \backsim 9C_4 - 2(A + B + E + F) - 6(I + J + M + N),\\
(CD)(GH)(KL)(OP).
\end{gathered}
$$

The periodicity of (8) depends on that of

$$C_4 \backsim 9C_4 - 2\alpha - 6\beta, \qquad \alpha \backsim 12C_4 - 3\alpha - 8\beta, \qquad \beta \backsim 4C_4 - 3\beta,$$

where $\alpha = A + B + E + F$ and $\beta = I + J + M + N$. Transform this through

$$X = C_4 - \beta, \qquad Y = 2C_4 - \alpha - \beta, \qquad Z = \beta,$$

and it becomes

$$Y \backsim Y, \qquad X \backsim 2Y + X, \qquad Z \backsim 4X + Z,$$

which is non-periodic. Hence (8) is non-periodic. It is to be noticed that the four vertices of each tetraedron of this product are invariant as a whole under the inversion taken with respect to the other tetraedron. The non-periodicity of the product of two inversions, taken with respect to any similarly chosen pair of Göpel even tetraedra, is an immediate consequence.

In a similar manner the transformation $[{AB \atop EF}][{JK \atop NO}]$ may be shown to be non-periodic. The distinction between this case and that immediately preceding is that $J\,K\,N\,O$ are not invariant as a whole under $[{AB \atop EF}]$ and $A\,B\,E\,F$ are not invariant as a whole under $[{JK \atop NO}]$. The non-periodicity of all similar products follows.

This completes the analysis of the product of two inversions taken with respect to tetraedra chosen in all possible positions. Hence follows the

THEOREM. *The product of any two cubic inversions of the general Kummer surface is periodic only if the Göpel even tetraedra, with respect to which the respective inversions are taken, have an edge in common.*

An immediate consequence of this is the theorem of Hutchinson* that the sixty inversions generate a group of infinite order.

* Loc. cit., p. 212.

Any line through a node of K_4 meets the surface in two other points. The transformation which interchanges these points is a monoidal cubic involution having the node for fundamental point. Under this nodal projection any plane section of K_4 is transformed into a C_{12} of genus three. Hence for the node A

$$C_4 \backsim 3C_4 - 4A. \tag{9}$$

The image of the node is the intersection of the surface K_4 with the tangent cone at the node, namely a C_8 of genus zero. Hence

$$A \backsim 2C_4 - 3A. \tag{10}$$

The other nodes remain invariant. Equations (9) and (10) are all that will be written to denote the complete nodal projection and will be indicated by $[A]$.

The product of two nodal projections $[B][A]$ gives

$$C_4 \backsim 9C_4 - 12A - 4B, \qquad A \backsim 2C_4 - 3A, \qquad B \backsim 6C_4 - 8A - 3B,$$

the rest of the nodes being invariant. This transformation may be shown to be non-periodic by the methods employed above.*

If a point be transformed by a cubic inversion and its image transformed by a nodal projection, the resulting transformation is either involutorial or non-periodic. For $[{}^{AB}_{EF}][A]$ gives

$$C_4 \backsim 5C_4 - 6A - 2(B + E + F), \qquad A \backsim 3C_4 - 4A - (B + E + F),$$
$$B \backsim C_4 - A - (E + F), \qquad E \backsim C_4 - A - (B + F),$$
$$F \backsim C_4 - A - (B + F), \qquad (CG)(DH)(IJ)(KP)(LO)(MN).$$

This is easily verified to be involutorial. If the node of projection is not a vertex of the tetraedron of inversion, every such product is of the same type as $[{}^{EF}_{AB}][C]$, under which

$$C_4 \backsim 9C_4 - 12C - 2(A + B + E + F),$$
$$A \backsim 3C_4 - 4C - (B + E + F), \qquad B \backsim 3C_4 - 4C - (A + E + F),$$
$$E \backsim 3C_4 - 4C - (A + B + F), \qquad F \backsim 3C_4 - 4C - (A + B + E),$$
$$G \backsim 2C_4 - 3C, \qquad C \backsim G, \qquad (DH)(IJ)(KP)(LO)(MN).$$

The non-periodicity of this transformation may be established as above. Hence *the product of a cubic inversion and a nodal projection is non-periodic except when the fundamental point of the projection is one of the basis points of the inversion.*

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* For another proof see V. Snyder, *An application of a* (1, 2) *quaternary correspondence to the Weddle and Kummer surfaces*, these Transactions, vol. 12 (1911), pp. 354–366; in particular, see p. 364.

2. The Weddle Surface

The Weddle surface W_4 can be obtained from the Kummer surface by birational transformation.* Under this transformation one node of the Kummer surface is transformed into a cubic curve on W_4. The six conics of K_4 through this node become the six nodes on the cubic curve on W_4. The remaining ten conics on K_4 are transformed into the ten lines of intersection of the pairs of planes determined by the six nodes on W_4. The other fifteen nodes of K_4 become the fifteen connecting lines of the six nodes of W_4. A plane section of K_4 is transformed into a curve of order eight on W_4.

In the diagram (1), if the cubic curve is taken as A, the residual elements will then be the fifteen lines joining the pairs of nodes on W_4. The six nodes will be denoted by b, c, d, e, i, and m; the remaining ten lines will be $a, f, g, h, j, k, l, n, o$, and p. These may be identified by the conics on K_4 to which they correspond. For example, through the node d pass the lines B, C, H, L, P, and the cubic curve A; the line k meets the lines C, G, I, J, L, and O.

Let variables x, y, z, w be so chosen that

$$e \equiv (1,0,0,0), \qquad i \equiv (0,1,0,0), \qquad m \equiv (0,0,1,0),$$

$$b \equiv (0,0,0,1), \qquad c \equiv (a,b,c,d), \qquad d \equiv (1/a, 1/b, 1/c, 1/d).$$

The equation of the Weddle surface, written in terms of these coördinates is (18) and is invariant under the transformation† (2). Under this transformation the pencils of planes through the edges of the tetraedron and the nodes c and d are interchanged in pairs. Hence the lines joining each vertex to c and d are interchanged. Thus

(12) $$(CD)\,(GH)\,(KL)\,(OP).$$

Under (2) an edge of the tetraedron of inversion is transformed into the opposite edge, or

(13) $$(EF)\,(IJ)\,(MN).$$

Further, the cubic curve A, through the six nodes, has the line B as image and conversely, thus

(14) $$(AB).$$

A quadric surface through the six nodes intersects W_4 in a curve C_8 of order eight which has a double point at each node and the lines $a, f, g, h, j, k, l, n, o$, and p for bisecants. Under (2) this curve is transformed into another

* The transformation here used is employed by Snyder, loc. cit., p. 360 and is the dual of the correspondence given by Hudson, loc. cit., p. 169.

† See Hutchinson, loc. cit., p. 216.

curve of the same system, since the quadric surface is transformed into another one through the six nodes; hence

$$(C_8 C_8). \tag{15}$$

It follows from (12), (13), (14), and (15) that the complete inversion is

$$(C_8 C_8)(AB)(CD)(EF)(GH)(IJ)(KL)(MN)(OP).$$

Since C_8 on W_4 corresponds to C_4 on K_4, it follows that this is a collineation on K_4. Hence *the collineation group on K_4 corresponds to the inversion group on W_4. On W_4 the product of two inversions is an inversion* and the inversion group is of order* 16.

Any line through a node of W_4 meets the surface in two other points. The transformation which interchanges these points is involutorial. Consider the projection N_c from the node c. The tangent cone at the node meets W_4 in a curve of order eight, which consists of the lines B, D, G, K, and O and the cubi A, hence (Ac). A plane through B intersects W_4 in B and a cubic curve. This curve intersects B in c, d, and a residual point R which is the image of d in this plane. As the plane rotates, R describes the line B. Hence (Bd), and similarly for the four remaining lines through c. Under N_c the line M is transformed into the ine of intersection of the planes Mc and Nd, and conversely. Hence (Mo), and similarly for all remaining lines. The collected result is

$$\begin{matrix}(Ac)(Bd)(Ca)(Db)(Eg)(Fh)(Ge)(Hf)\\(Ik)(Jl)(Ki)(Lj)(Mo)(Np)(Om)(Pn).\end{matrix} \tag{16}$$

Since $C_8 = 2a + B + C + D + E + I + M$, it follows from (16) that $C_8 \backsim 2C + d + a + b + g + k + o$. By writing the symbolic sums for C_8 which contain the elements a, b, d, g, k, o, these may be expressed in terms of C_8 and the elements $A, B, \cdots, P$. Hence

$$C_8 \backsim 3C_8 - (A + B + \cdots + P). \tag{17}$$

The transformation N_c is given by (16) and (17). In like manner the transformation N_d is given by (17) and

$$\begin{matrix}(Ad)(Bc)(Cb)(Da)(Eh)(Fg)(Gf)(He)\\(Il)(Jk)(Kj)(Li)(Mp)(No)(On)(Pm).\end{matrix}$$

* In Hutchinson's paper, loc. cit., p. 217, the product of two inversions is stated to be of infinite order. The formulas used for changing the fundamental tetraedron from $e\ i\ m\ b$ to $e\ i\ m\ c$ should read, in his notation,

$$w_1 = w, \quad \sqrt{b^2 - a^2}\, x_1 = bw - ax, \quad \sqrt{c^2 - a^2}\, y_1 = cw - ay, \quad \sqrt{d^2 - a^2}\, z_1 = dw - az.$$

With this change it is seen that the product of the two is an inversion.

The product of these transformations is the inversion with respect to the remaining nodes. Hence *the cubic inversion with respect to any tetraedron of nodes on the Weddle surface is the product of the nodal projections with respect to the two residual nodes.**

The points (x, y, z, w), $(1/x, 1/y, 1/z, 1/w)$, (a, b, c, d), and $(1/a, 1/b, 1/c, 1/d)$ are coplanar if

$$\begin{vmatrix} x & y & z & w \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} & \frac{1}{w} \\ a & b & c & d \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \end{vmatrix} = 0. \tag{18}$$

This condition is satisfied if (x, y, z, w) is on the Weddle surface. In other words, a point on W_4 and its image under a cubic inversion are coplanar with the residual nodes of the tetraedron of inversion. Thus any plane through the residual nodes is invariant under an inversion, as is well known for the equivalent product of the two nodal projections.

Let the nodal projections be designated by N_i, $i = 1, 2, \cdots, 6$. The inversions will be $N_i N_k = I_{ik}$. Hence $N_i = I_{ik} N_k$. There results immediately the known theorem that the group of the nodal projections is of order 32. For it is the direct product of the inversion group of order 16 by the group generated by any nodal projection.

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* The algebraical equivalent of this theorem is given by Baker, *Multiply Periodic Functions*, p. 154.